Signal Analysis & Communication ECE355

Ch. 4.1: CTFT (contd.)

Ch. 4.2: FT of Periodic Signals.

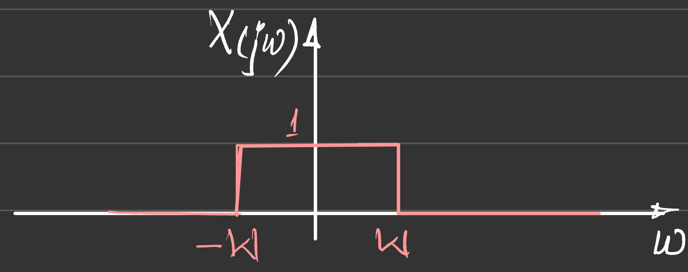
Lecture 19

23-10-2023

Ch. 4.1 CTFT (contd.)

Example 3

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$



"IDEAL LOW PASS FILTER"

$x(t) = ?$

using synthesis eqn of CTFT

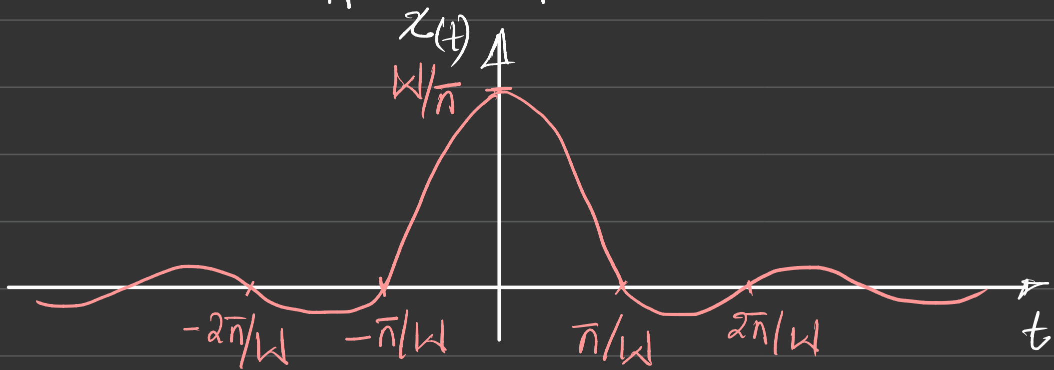
$$x(t) = \frac{1}{2\pi} \int_{-W}^{W} (1)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \times \frac{1}{jt}\, e^{j\omega t} \Big|_{-W}^{W}$$

$$= \frac{1}{\pi t} \left[\frac{e^{jWt} - e^{-jWt}}{j2} \right]$$

$$= \frac{\sin Wt}{\pi t} = \frac{W}{\pi} \times \frac{\sin \pi (Wt/\pi)}{\pi (Wt/\pi)}$$

$$= \frac{W}{\pi}\, \mathrm{sinc}\left(\frac{W}{\pi} t\right)$$



NOTE:

DUALITY
- Pulse in time domain ⟷ Sinc function in freq. domain (Example 2 – Lecture 18)
- Sinc function in time domain ⟷ Pulse in freq. domain (Example 3 – Lecture 19 this lecture!)

we'll do later!

Example 4

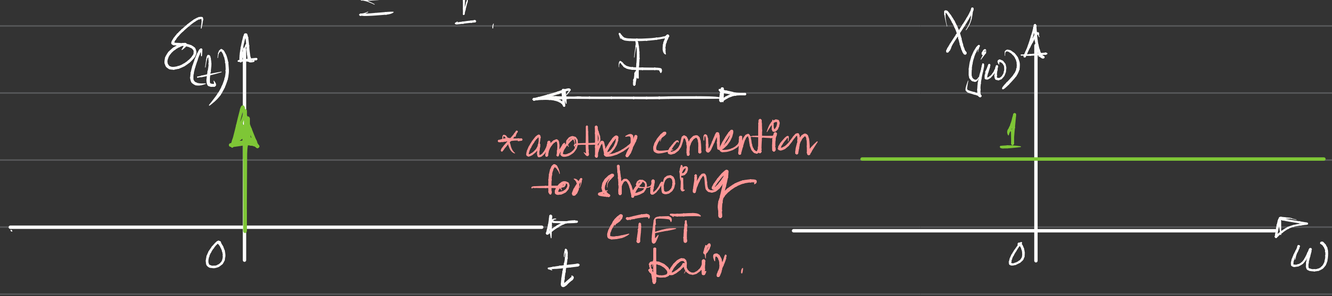
$x(t) = \delta(t)$, $X(j\omega) = ?$

Using Analysis eqn. of CTFT

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega(0)}\,(1)$$

$$= 1.$$



Example 5

$X(j\omega) = 2\pi\,\delta(\omega)$, $x(t) = ?$

using Synthesis eqn of CTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\,\delta(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi} \times 2\pi\, e^{j(0)t}\,(1)$$

$$= 1$$

<u>NOTE</u> If we do it conversely, i.e., for $x(t) = 1$, $\forall t$

$$X(j\omega) = \underbrace{\int_{-\infty}^{\infty} (1)\, e^{-j\omega t}\, dt}_{\text{Does not converge under "Basic" Calculus!}} = 2\pi\,\delta(\omega)$$

NOTE

DUALITY

Impulse in time domain $\longleftrightarrow$ Constant DC in freq. domain (Example 4)

Constant DC in time domain $\longleftrightarrow$ Impulse in freq. in domain (Example 5)

Example 6

$$x(t) = u(t) \,,\quad X(j\omega) = ?$$

Using Analysis eqn. of CTFT

$$X(j\omega) = \int_{-\infty}^{\infty} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-j\omega t}\, dt$$

Doesnot converge
* May contain δ function!
(Example 5)

* Due to dicontinuity of $u(t)$ at $t=0$

Fact $\boxed{X(j\omega) = \frac{1}{j\omega} + \pi\,\delta(\omega)}$

So we have

$$u(t) \longleftrightarrow \frac{1}{j\omega} + \pi\,\delta(\omega)$$

Proof (Conversely) OPTIONAL

- Using Synthesis eqn. of CTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{j\omega} + \pi\,\delta(\omega) \right) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{j\omega} e^{j\omega t} d\omega$$

$$= \frac{1}{2} + \frac{1}{2\pi} \left(\int_{0}^{\infty} \frac{1}{j\omega} e^{j\omega t} d\omega + \int_{0}^{\infty} -\frac{1}{j\omega} e^{-j\omega t} d\omega \right)$$

$$= \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin(\omega t)}{\omega} d\omega$$

$$\overset{\omega' = \omega t}{=} \begin{cases} \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin(\omega')}{\omega'} d\omega' , & t > 0 \\ \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \frac{\sin(\omega')}{\omega'} d\omega' , & t < 0 \end{cases}$$

$$= \begin{cases} 1 , & t > 0 \\ 0 , & t < 0 \end{cases}$$

$$= u(t)$$

Ch. 4.2: FOURIER TRANSFORMS OF PERIODIC SIGNALS

- We can develop FT representation for periodic signals, which allows us to consider both periodic & aperiodic signals within a unified context

* The resulting transform consists of a train of impulses in the freq. domain, with the areas of impulses proportional to the FS coefficients.

- For having the general result, let us first consider $x(t)$ with FT $X(j\omega)$, which is a single impulse of area 2π at $\omega = \omega_0$; that is,

$$X(j\omega) = 2\pi \delta(\omega - \omega_0) \quad — ①$$

$\mathcal{F}$

- Let's apply synthesis eqn. of FT for finding $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi \delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t} \quad — ②$$

- Now we that the periodic sig. can be expressed as:

$$x_{(t)} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

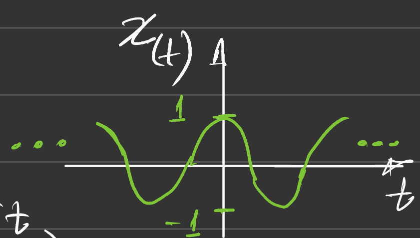
- Eqn ① & Eqn ② implies that

$$X_{(j\omega)} = \sum_{k=-\infty}^{\infty} 2\pi a_k \underbrace{\delta_{(\omega - k\omega_0)}}_{\text{Impulse exists at } \omega = k\omega_0 \text{ where } k: -\infty \to \infty} \quad — ③$$

Train of impulses occuring at the harmonically related frequencies * above!

Example 1

$$x_{(t)} = \cos(2\pi t)$$

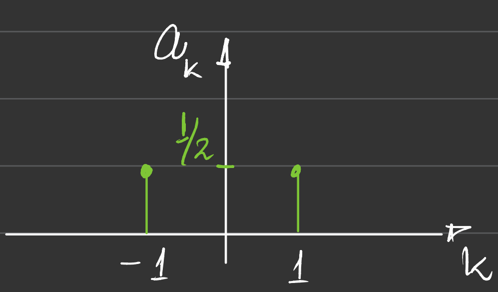


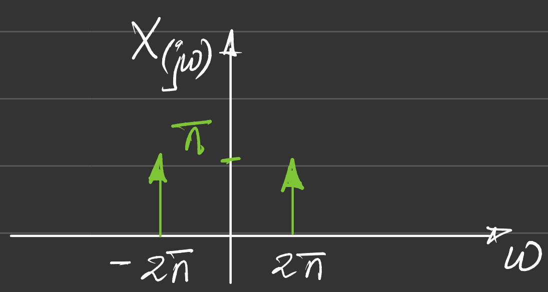
- We know CTFS of $x_{(t)} = \frac{1}{2}\left(e^{j2\pi t} + e^{-j2\pi t}\right)$

with $\omega_0 = 2\pi$, $T = 1$.

$$a_1 = a_{-1} = \frac{1}{2}$$

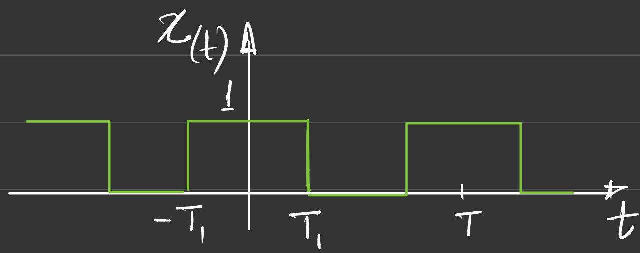
$$a_k = 0 \qquad \text{for } k \neq \pm 1$$





$$X_{(j\omega)} = 2\pi \times \frac{1}{2}\left[\delta_{(\omega - 2\pi)} + \delta_{(\omega + 2\pi)}\right]$$

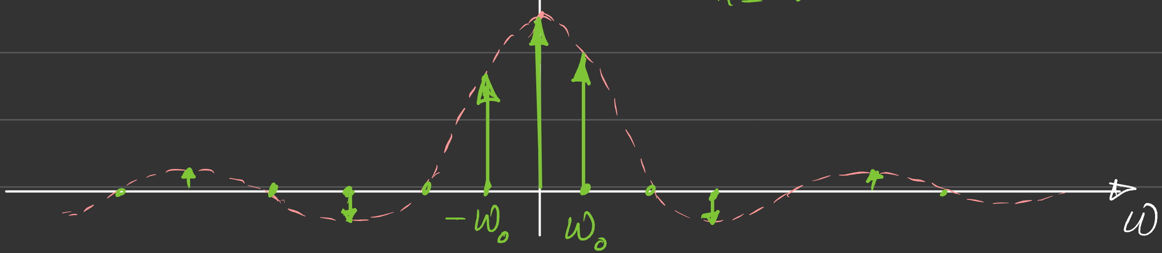
Example 2



$$\xleftrightarrow{FS} \quad a_k = \begin{cases} \dfrac{\sin k\pi/2}{k\pi} & , \; k \neq 0 \\ 1/2 & , \; k = 0 \end{cases}$$

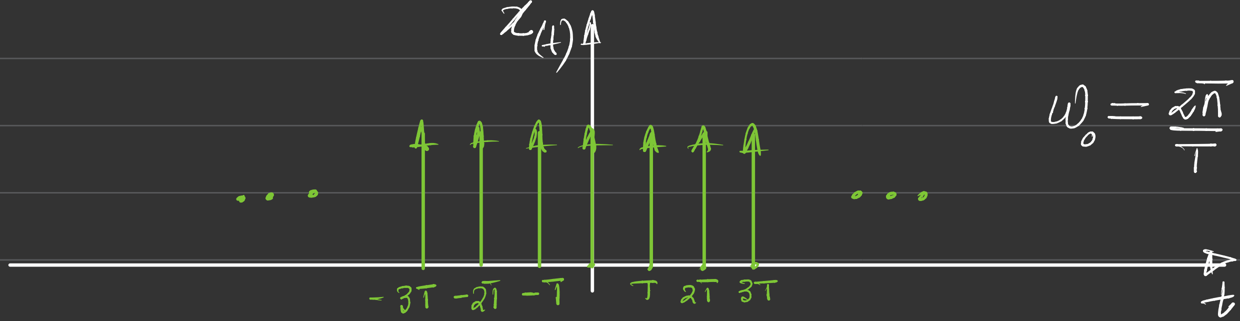
CTFT of this sig. is :
(using eqn ③)

$$X_{(j\omega)} = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0)$$



Example 3

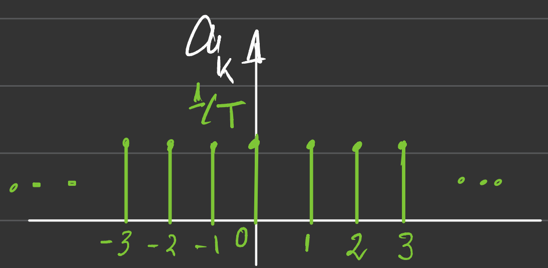
$$x_{(t)} = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$



$$\omega_0 = \frac{2\pi}{T}$$

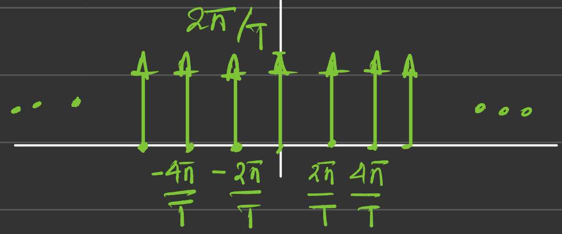
$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta_{(t)} \, e^{-jk\omega_0 t} \, dt$$

$$= \frac{1}{T} e^{0} (1) = \frac{1}{T}$$



$\longrightarrow$

$$X_{(j\omega)} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$$



NOTE: Impulse train in time domain $\longleftrightarrow$ Impulse train in freq. domain